Paraxial Helmholtz Equation & the Transverse Laplacian

**Paraxial Helmholtz Equation**

"Consider a light beam with complex amplitude \( U(r) = A(r) \exp(-ikz) \) which is close to being a plane wave, i.e. \( A(r) \) is a function of position which varies very slowly on a distance scale of a wavelength. By making the 'slow-varying envelope' approximation, i.e. assuming that \( \partial A / \partial z \ll kA \) and \( \partial^2 A / \partial z^2 \ll k^2 A \), show that the Helmholtz equation becomes \( \nabla^2 A - 2ik \partial A / \partial z = 0 \), where \( \nabla^2 = \nabla_x \cdot \nabla_y \) is the 'transverse Laplacian operator'. This form \( \nabla^2 A - 2ik \partial A / \partial z = 0 \) is known as the 'paraxial Helmholtz equation'.

The beam complex amplitude is given as:

\[ U(r) = A(r) e^{-ikz} \]

Additionally, the Helmholtz equation is:

\[ \nabla^2 U + k^2 U = 0 \]

By expanding the Cartesian Laplacian and factorising:

\[ (\partial_x^2 + \partial_y^2 + \partial_z^2 + k^2)U = 0 \]

Collect transverse terms (subscript on a function denotes partial derivative):

\[ (\partial_x^2 + \partial_y^2)U + (\partial_z^2 + k^2)U = 0 \]

\[ \nabla_x^2 U + U_{zz} + k^2 U = 0 \]

Substitute expression for \( U \) and evaluate:

\[ \nabla_x^2 A e^{-ikz} + \frac{\partial^2}{\partial z^2} \left( A e^{-ikz} \right) + k^2 A e^{-ikz} = 0 \]

\[ \nabla_x^2 A e^{-ikz} + \nabla^2 \left( A e^{-ikz} - ikA e^{-ikz} \right) + k^2 A e^{-ikz} = 0 \]

\[ \nabla_x^2 A e^{-ikz} + \left( A_x e^{-ikz} - k^2 A e^{-ikz} - 2ikA_x e^{-ikz} \right) + k^2 e^{-ikz} = 0 \]

Remove complex (exponential) factors and cancel "\( k^2 \)" parts:

\[ \nabla_x^2 A + A_{zz} - 2ikA_x = 0 \]

As \( |2ikA_x| = 2k|A_x| \), and we are given that \( \frac{\partial^2 A}{\partial z^2} \ll k \frac{\partial A}{\partial z} \), we deduce \( |A_{zz}| \ll |2ikA_x| \), allowing removal of the \( A_{zz} \) term.

This gives the paraxial Helmholtz equation:

\[ \nabla_x^2 A - 2ik \frac{\partial A}{\partial z} = 0 \]
Solution to the paraxial Helmholtz equation

“Consider a wave with complex amplitude of the general form \( U(\mathbf{r}) = \frac{A_h}{q(z)} e^{-\frac{i(x^2+y^2)}{2q(z)}} e^{-ikz} \) where \( q(z) = z + iz_o \) and \( z_o \) is a constant. Show that this is a solution of the paraxial Helmholtz equation.”

\[ q(z) = z + iz_o \]

\[ U(\mathbf{r}) = \frac{A_h}{q(z)} e^{-\frac{i(x^2+y^2)}{2q(z)}} e^{-ikz} = A(\mathbf{r}) e^{-ikz} \]

i.e. \( A(\mathbf{r}) = \frac{A_h}{q(z)} e^{-\frac{i(x^2+y^2)}{2q(z)}} \)

First, compact the equation:

\( B = -ik; \quad r^2 = x^2 + y^2; \quad f(z) = \frac{1}{q(z)} \)

\[ \therefore A = A_h f e^{\frac{ik}{2f}} \]

The paraxial Helmholtz equation in Cartesian co-ordinates is:

\[ \nabla_\perp^2 A - 2ik \frac{\partial A}{\partial z} = 0 \]

Firstly, evaluate the transverse part (\( \nabla_\perp^2 A \)) of the equation:

\[ A_{xx} = AB f(1 + Bf x^2); \quad \text{Similarly for } y: \quad A_{yy} = AB f(1 + Bf y^2) \]

\[ \therefore \nabla_\perp^2 A = A_{xx} + A_{yy} = ABf \left[ 2 + 2Bf(x^2 + y^2) \right] = ABf \left[ 2 + B^2f \right] \]

Next, evaluate the axial part (-2ikA_z) of the equation, using the chain rule:

\[ \frac{\partial A}{\partial z} = \frac{\partial A}{\partial f} \frac{\partial f}{\partial q} \frac{\partial q}{\partial z} = -1 \frac{\partial A}{q^2} = -f^2 \frac{\partial A}{\partial f} \]

\[ -2ik \frac{\partial A}{\partial z} = 2ik f^2 \frac{\partial A}{\partial f} \]

\[ = -2Bf^2 \frac{\partial A}{\partial f} \]

\[ = -2Bf \left[ A_h e^{\frac{ik}{2f}} + A_h \frac{B}{2f} e^{\frac{ik}{2f}} \right] \]

\[ = -2ABf \left[ \frac{1}{f} + \frac{B^2}{2} \right] \]

\[ = -ABf \left[ 2 + B^2 f \right] \]

Finally, sum the two parts together:

\[ \nabla_\perp^2 A - 2ik \frac{\partial A}{\partial z} = ABf \left[ 2 + B^2 f \right] + -ABf \left[ 2 + B^2 f \right] \]

\[ = 0 \]

\( U(\mathbf{r}) \) is indeed a solution of the paraxial Helmholtz equation.