Paraxial Helmholtz Equation & the Transverse Laplacian

Paraxial Helmholtz Equation

"Consider a light beam with complex amplitude $U(\mathbf{r}) = A(\mathbf{r})\exp(-ikz)$ *which is close to being a plane wave, i.e.* $A(\mathbf{r})$ *is a function of position which varies very slowly on a distance scale of a wavelength. By making the 'slow-varying envelope' approximation, i.e. assuming that* $\partial A/\partial z << kA$ *and* $\partial^2 A/\partial z^2 << k.\partial A/\partial z$*, show that the Helmholtz equation becomes* $\nabla_T^2 A - 2ik.\partial A/\partial z = 0$*, where* $\nabla_T = \partial_{xx} + \partial_{yy}$ *is the 'transverse Laplacian operator'. This form* $\nabla_T^2 A - 2ik.\partial A/\partial z = 0$ *is known as the 'paraxial Helmholtz equation'.*

The beam complex amplitude is given as:

$$U(\mathbf{r}) = A(\mathbf{r})e^{-ikz}$$

Additionally, the helmholtz equation is:

$$\nabla^2 U + k^2 U = 0$$

By expanding the Cartesian Laplacian and factorising:

$$(\partial_x^2 + \partial_y^2 + \partial_z^2 + k^2)U = 0$$

Collect transverse terms (subscript on a function denotes partial derivative):

$$(\partial_x^2 + \partial_y^2)U + (\partial_z^2 + k^2)U = 0$$

$$\nabla_T^2 U + U_{zz} + k^2 U = 0$$

Substitute expression for U and evaluate:

$$\nabla_T^2 A e^{-ikz} + \frac{\partial^2}{\partial z^2}\left(Ae^{-ikz}\right) + k^2 A e^{-ikz} = 0$$

$$\nabla_T^2 A e^{-ikz} + \frac{\partial}{\partial z}\left(A_z e^{-ikz} - ikAe^{-ikz}\right) + k^2 A e^{-ikz} = 0$$

$$\nabla_T^2 A e^{-ikz} + (A_{zz} e^{-ikz} - k^2 A e^{-ikz} - 2ikA_z e^{-ikz}) + k^2 e^{-ikz} = 0$$

Remove complex (exponential) factors and cancel "k^2" parts:

$$\nabla_T^2 A + A_{zz} - 2ikA_z = 0$$

As $\left|-2ikA_z\right| = 2\left|kA_z\right|$, and we are given that $\frac{\partial^2 A}{\partial z^2} << k\frac{\partial A}{\partial z}$, we deduce $\left|A_{zz}\right| << \left|-2ikA_z\right|$, allowing removal of the A_{zz} term.

This gives the paraxial Helmholtz equation:

$$\nabla_T^2 A - 2ik\frac{\partial A}{\partial z} = 0$$

Solution to the paraxial Helmholtz equation

"Consider a wave with complex amplitude of the general form $U(\mathbf{r}) = \frac{A_0}{q(z)} e^{-ikz} e^{-ik\frac{x^2+y^2}{2q(z)}}$ *where* $q(z) = z + iz_0$ *and* z_0 *is a constant. Show that this is a solution of the paraxial Helmholtz equation."*

$$\mathrm{q}(z) = z + iz_0$$

$$U(\mathbf{r}) = \frac{A_0}{q(z)} e^{-ik\frac{x^2+y^2}{2q(z)}} e^{-ikz} = A(\mathbf{r})e^{-ikz}$$

i.e. $A(\mathbf{r}) = \frac{A_0}{q(z)} e^{-ik\frac{x^2+y^2}{2q(z)}}$

First, compact the equation:

$$B = -ik;\ r^2 = x^2 + y^2;\ \mathrm{f}(z) = 1/\mathrm{q}(z)$$

$$\therefore A = A_0 f e^{\frac{B}{2}r^2 f}$$

The paraxial Helmholtz equation in Cartesian co-ordinates is:

$$\nabla_T^2 A - 2ik\frac{\partial A}{\partial z} = 0$$

Firstly, evaluate the transverse part ($\nabla_T^2 A$) of the equation:

$A_{xx} = AB\mathrm{f}(1 + B\mathrm{f}\ x^2)$; Similarly for y: $A_{yy} = AB\mathrm{f}(1 + B\mathrm{f}\ y^2)$

$$\therefore \nabla_T^2 A = A_{xx} + A_{yy} = ABf\left[2 + B\mathrm{f}(x^2 + y^2)\right] = ABf\left[2 + Br^2\,\mathrm{f}\right]$$

Next, evaluate the axial part (-2ikA$_z$) of the equation, using the chain rule:

$$\frac{\partial A}{\partial z} = \frac{\partial A}{\partial f}\frac{\partial f}{\partial q}\frac{\partial q}{\partial z} = \frac{-1}{\mathrm{q}^2}\frac{\partial A}{\partial \mathrm{f}}1 = -\mathrm{f}^2\frac{\partial A}{\partial \mathrm{f}}$$

$$\begin{aligned}
-2ik\frac{\partial A}{\partial z} &= 2ik\,\mathrm{f}^2\frac{\partial A}{\partial \mathrm{f}} \\
&= -2B\mathrm{f}^2\frac{\partial A}{\partial \mathrm{f}} \\
&= -2B\mathrm{f}^2\left[A_0 e^{\frac{B}{2}r^2\mathrm{f}} + A_0\frac{B}{2}r^2\,\mathrm{f}\,e^{\frac{B}{2}r^2\mathrm{f}}\right] \\
&= -2AB\mathrm{f}^2\left[\frac{1}{\mathrm{f}} + \frac{Br^2}{2}\right] \\
&= -AB\mathrm{f}\left[2 + Br^2\,\mathrm{f}\right]
\end{aligned}$$

Finally, sum the two parts together:

$$\begin{aligned}
\nabla_T^2 A - 2ik\frac{\partial A}{\partial z} &= AB\mathrm{f}\left[2 + Br^2\,\mathrm{f}\right] + -AB\mathrm{f}\left[2 + Br^2\,\mathrm{f}\right] \\
&= 0
\end{aligned}$$

$U(\mathbf{r})$ is indeed a solution of the paraxial Helmholtz equation.